

Two-particle Irreducible Effective Action Approach to Correlated Electron Systems

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Abstract

The two-particle irreducible (2PI) effective action theories are employed to study the strongly fluctuating electron systems, under the formalism of the two-dimensional Hubbard model. We obtain the corresponding quantum 2PI effective action after the original classic action of the Hubbard model is bosonized. In our actual calculations, the 2PI effective action is expanded to three loops, in which the leading order (LO) and next-to-leading order (NLO) quantum fluctuations are included. Numerical calculations indicate that the NLO fluctuations should not be neglected when the Coulomb on-site repulsion energy is larger than two times the nearest-neighbor hopping energy.

Keywords:

Two-particle irreducible effective action, Strongly correlated electron systems, Self-consistent equations

1. Introduction

Since the high-temperature superconductivity was discovered in 1986 [1], It has been believed that the appropriate model to describe the strongly correlated electron systems is the nearly half-filled two-dimensional Hubbard model with moderately large repulsion energy U and antiferromagnetic exchange constant $J = 4t^2/U$ where t is the site hopping [2]. The Hubbard model is just composed of two terms: one is the site-hopping term which forms the band structure, and the other is the Coulomb repulsion term that

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represents the interaction between electrons. Therefore, from its appearance, it looks like that the two-dimensional Hubbard model is a simple model and it is easy to solve. However, the actual situations are quite different. The Hubbard model describes a many-body electron system in which the interacting potential energy and the kinetic energy are comparable. So we cannot employ the perturbation theory and treat the potential energy or the kinetic energy as a perturbation.

Lots of efforts have been made to solve the two-dimensional Hubbard model and many methods have been developed. For example, the quantum Monte Carlo simulations [3], the self-consistent approach of conserving approximations [4], the variational cluster perturbation theory [5], the functional renormalization group approach [6], and so on. In recent years, another nonperturbative approach, known as the two-particle irreducible (2PI) effective action theory first introduced in the field theory [7], has attracted lots of attentions. The 2PI effective action theory resums certain classes of diagrams to infinite order, so nonperturbative effects are included in this approach. Furthermore, in the 2PI formalism, the effective action can be expanded according to the order of the loop or $1/N$ in the $O(N)$ model. Therefore, it is easy to investigate the effects of the high order contributions in the 2PI effective action theory. In the studies of field theories, it has been found that the 2PI effective action theory is very successful in describing equilibrium thermodynamics, and also the quantum dynamics of far from equilibrium of quantum fields. The entropy of the quark-gluon plasma obtained in the 2PI formalism shows very good agreement with lattice data for temperatures above twice the transition temperature [8]. The poor convergence problem usually encountered in high-temperature resummed perturbation theory with bosonic fields is also solved in the 2PI effective action theory [9]. Furthermore, it has been shown that non-equilibrium dynamics with subsequent late-time thermalization can be well described in the 2PI formalism (see [10] and references therein). The 2PI effective action has also been combined with the exact renormalization group to provide efficient non-perturbative approximation schemes [11]. The shear viscosity in the $O(N)$ model has been computed using the 2PI formalism [12]. Specially, we would like to emphasize that due to many people's contributions [13, 14, 15, 16], it has been clear that the 2PI effective action theory can be renormalized, which is quite non-trivial for a non-perturbative approach.

In this work, we will employ the 2PI effective action theory to investigate the strong fluctuations of electron systems. We will expand the effective

action to three loops and compute the leading order (LO) and the next-to-leading order (NLO) contributions to the fermion and boson self-energies. Then we will investigate when the importance of the high order quantum fluctuations becomes significant with the increase of the Coulomb repulsion energy U . The paper is organized as follows. In section 2 we apply the 2PI formalism into the Hubbard model and obtain its effective action. In section 3 we obtain the LO and NLO contributions to the fermion and boson self-energies. Numerical results are presented in Sec. 4. In section 5 we give our summary and conclusions.

2. 2PI Effective Action Theory

We begin with the simplest two-dimensional one-band Hubbard model which reads

$$H = -t \sum_{\langle ij \rangle \sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + U \sum_i \hat{c}_{i\uparrow}^\dagger \hat{c}_{i\uparrow} \hat{c}_{i\downarrow}^\dagger \hat{c}_{i\downarrow}, \quad (1)$$

where we only consider the nearest-neighbor hopping t and U is the Hubbard on-site Coulomb repulsion energy. We rewrite the interaction term as

$$\hat{c}_{i\uparrow}^\dagger \hat{c}_{i\uparrow} \hat{c}_{i\downarrow}^\dagger \hat{c}_{i\downarrow} = -\frac{1}{2}(\hat{c}_i^\dagger \sigma^z \hat{c}_i)^2 + \frac{1}{2}(\hat{c}_{i\uparrow}^\dagger \hat{c}_{i\uparrow} + \hat{c}_{i\downarrow}^\dagger \hat{c}_{i\downarrow}) \quad (2)$$

for the convenience of calculations below, where σ^z is the z component of the Pauli matrices. The second term on the right hand side of Eq. (2) can be absorbed in the chemical potential term. Then we arrive at

$$H - \mu N = -t \sum_{\langle ij \rangle \sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} - \frac{U}{2} \sum_i (\hat{c}_i^\dagger \sigma^z \hat{c}_i)^2 - \mu \sum_{i\sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{i\sigma}. \quad (3)$$

The classic action corresponding to Hamiltonian given above is

$$S = \int dt \left[\sum_{i\sigma} c_{i\sigma}^* i \partial_t c_{i\sigma} + t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^* c_{j\sigma} + \mu \sum_{i\sigma} c_{i\sigma}^* c_{i\sigma} + \frac{U}{2} \sum_i (c_i^\dagger \sigma^z c_i)^2 \right], \quad (4)$$

where creating and annihilating operators in Eq. (3) are replaced by their Grassmann fields. Including quantum and thermal fluctuations, one obtains the generating functional, also known as the partition function, which reads

$$Z[\eta^*, \eta, J] = \int [dc^*][dc] \exp \left\{ - \int_0^\beta d\tau \left[\sum_{i\sigma} c_{i\sigma}^* \partial_\tau c_{i\sigma} - t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^* c_{j\sigma} \right. \right.$$

$$\begin{aligned}
& -\mu \sum_{i\sigma} c_{i\sigma}^* c_{i\sigma} - \frac{U}{2} \sum_i (c_i^\dagger \sigma^z c_i)^2 + \sum_{i\sigma} (\eta_{i\sigma}^* c_{i\sigma} + c_{i\sigma}^* \eta_{i\sigma}) \\
& + \sum_i J_i (-1)^i c_i^\dagger \sigma^z c_i \Big] \Big\}. \tag{5}
\end{aligned}$$

We will employ the Matsubara imaginary-time formalism throughout this work and here $\beta = 1/T$ is the inverse of the temperature. An external source term for the composite operator $(-1)^i c_i^\dagger \sigma^z c_i$ is included in Eq. (5).

Before we continue the calculations, it would be more convenient if the bosonization of the action in Eq. (5) is made first. Up to a constant, we have

$$\begin{aligned}
& \exp \left\{ - \int_0^\beta d\tau \left[- \frac{U}{2} \sum_i (c_i^\dagger \sigma^z c_i)^2 + \sum_i J_i (-1)^i c_i^\dagger \sigma^z c_i \right] \right\} \\
= & \exp \left[- \frac{1}{2U} \int_0^\beta d\tau \sum_i J_i^2 \right] \int [dB] \exp \left\{ - \int_0^\beta d\tau \sum_i \left[\frac{U}{2} B_i^2 \right. \right. \\
& \left. \left. - U B_i (-1)^i c_i^\dagger \sigma^z c_i + J_i B_i \right] \right\}, \tag{6}
\end{aligned}$$

where a boson field B is introduced through its functional integral. Substituting Eq. (6) into Eq. (5) and neglecting the irrelevant prefactor on the right hand side of Eq. (6), one finds

$$\begin{aligned}
Z[\eta^*, \eta, J] = & \int [dc^*][dc][dB] \exp \left\{ - \int_0^\beta d\tau \left[\sum_{i\sigma} c_{i\sigma}^* \partial_\tau c_{i\sigma} - t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^* c_{j\sigma} \right. \right. \\
& - \mu \sum_{i\sigma} c_{i\sigma}^* c_{i\sigma} + \sum_i \left(\frac{U}{2} B_i^2 - U B_i (-1)^i c_i^\dagger \sigma^z c_i \right) \\
& \left. \left. + \sum_{i\sigma} (\eta_{i\sigma}^* c_{i\sigma} + c_{i\sigma}^* \eta_{i\sigma}) + \sum_i J_i B_i \right] \right\}. \tag{7}
\end{aligned}$$

In order to obtain the 2PI effective action for the Hubbard model, we need add two-point sources into Eq. (7). Then we obtain

$$\begin{aligned}
Z[\eta^*, \eta, J, M, K] = & \int [dc^*][dc][dB] \exp \left\{ - [I_0(c^*, c) + I_0(B) + I_{\text{int}}(c^*, c, B) \right. \\
& \left. + \eta^* c + c^* \eta + JB + \frac{1}{2} BMB + c^* Kc] \right\}, \tag{8}
\end{aligned}$$

where the two-point external sources are

$$\frac{1}{2}BMB \equiv \frac{1}{2} \int_0^\beta d\tau_i d\tau_j \sum_{ij} B_i(\tau_i) M_{ij}(\tau_i, \tau_j) B_j(\tau_j), \quad (9)$$

$$c^* K c \equiv \int_0^\beta d\tau_i d\tau_j \sum_{ij} \sum_{\alpha\beta} c_{i\alpha}^*(\tau_i) K_{i\alpha, j\beta}(\tau_i, \tau_j) c_{j\beta}(\tau_j). \quad (10)$$

Here α and β are spin indices. In Eq. (8) we also used the following abbreviated notations:

$$I_0(c^*, c) \equiv \int_0^\beta d\tau \left[\sum_{i\alpha} c_{i\alpha}^* \partial_\tau c_{i\alpha} - t \sum_{\langle ij \rangle \alpha} c_{i\alpha}^* c_{j\alpha} - \mu \sum_{i\alpha} c_{i\alpha}^* c_{i\alpha} \right], \quad (11)$$

$$I_0(B) \equiv \int_0^\beta d\tau \sum_i \frac{U}{2} B_i^2, \quad (12)$$

$$I_{\text{int}}(c^*, c, B) \equiv \int_0^\beta d\tau \sum_i (-U) B_i (-1)^i c_i^\dagger \sigma^z c_i, \quad (13)$$

$$\eta^* c + c^* \eta \equiv \int_0^\beta d\tau \sum_{i\alpha} (\eta_{i\alpha}^* c_{i\alpha} + c_{i\alpha}^* \eta_{i\alpha}), \quad (14)$$

$$JB \equiv \int_0^\beta d\tau \sum_i J_i B_i. \quad (15)$$

Introducing the generating functional for the connected Green functions

$$W[\eta^*, \eta, J, M, K] = -\ln Z[\eta^*, \eta, J, M, K], \quad (16)$$

it then follows that

$$\frac{\delta W}{\delta J_i} = B_i^c, \quad \frac{\delta W}{\delta \eta_{i\alpha}^*} = c_{i\alpha}^c, \quad \frac{\delta W}{\delta \eta_{i\alpha}} = -c_{i\alpha}^{*c}, \quad (17)$$

$$\frac{\delta W}{\delta M_{ji}} = \frac{1}{2}(B_i^c B_j^c + G_{ij}), \quad \frac{\delta W}{\delta K_{j\beta, i\alpha}} = -(c_{i\alpha}^c c_{j\beta}^{*c} + S_{i\alpha, j\beta}), \quad (18)$$

where B^c , c^c , and c^{*c} are the expected values of fields B , c , and c^* , respectively. G and S are the propagators for boson and fermion fields.

The 2PI effective action can be obtained from W through the Legendre transformation as follows

$$\Gamma[c^c, c^{*c}, B^c, G, S] = W[\eta^*, \eta, J, M, K] - J_i \frac{\delta W}{\delta J_i} - \eta_{i\alpha}^* \frac{\delta W}{\delta \eta_{i\alpha}^*} - \eta_{i\alpha} \frac{\delta W}{\delta \eta_{i\alpha}}$$

$$\begin{aligned}
& -M_{ji} \frac{\delta W}{\delta M_{ji}} - K_{j\beta, i\alpha} \frac{\delta W}{\delta K_{j\beta, i\alpha}} \\
& = W[\eta^*, \eta, J, M, K] - J_i B_i^c - \eta_{i\alpha}^* c_{i\alpha}^c - c_{i\alpha}^{*c} \eta_{i\alpha} \\
& \quad - \frac{1}{2} \text{Tr}[M(B^c B^c + G)] + \text{Tr}[K(c^c c^{*c} + S)], \quad (19)
\end{aligned}$$

where summations and integrals are assumed for the repeated indices. The trace operates in the coordinate and inner spaces. It can be easily proved that

$$\frac{\delta \Gamma}{\delta B_i^c} = -J_i - M_{ij} B_j^c, \quad \frac{\delta \Gamma}{\delta c_{i\alpha}^{*c}} = -\eta_{i\alpha} - K_{i\alpha, j\beta} c_{j\beta}^c, \quad (20)$$

$$\frac{\delta \Gamma}{\delta c_{i\alpha}^c} = \eta_{i\alpha}^* + c_{j\beta}^{*c} K_{j\beta, i\alpha}, \quad \frac{\delta \Gamma}{\delta G_{ij}} = -\frac{1}{2} M_{ji}, \quad \frac{\delta \Gamma}{\delta S_{i\alpha, j\beta}} = K_{j\beta, i\alpha}. \quad (21)$$

Equations (20) and (21) form a set of self-consistent equations which determine the field expected values B^c , c^c , c^{*c} and the propagators G and S , if the effective action can be expressed as a functional of these field expected values and propagators. Usually, the expected values of fermion field c^c and c^{*c} are vanishing when their external sources η^* and η are zero. We will assume $c^c = c^{*c} = 0$ in the following calculations and use B in place of B^c without confusions. It can be shown that the 2PI effective action can be expressed as [7]

$$\begin{aligned}
\Gamma(B, G, S) &= I(B) + \frac{1}{2} \text{Tr} \ln G^{-1} + \frac{1}{2} \text{Tr}(G_0^{-1} G) \\
&\quad - \text{Tr} \ln S^{-1} - \text{Tr}(S_0^{-1} S) + \Gamma_{\text{int}}(B, G, S), \quad (22)
\end{aligned}$$

with

$$\Gamma_{\text{int}}(B, G, S) = -\text{Tr}[(-\Sigma_{\text{mean}})S] + \Gamma_2(G, S). \quad (23)$$

Here we have $I(B) = I_0(B)$ and

$$G_0^{-1} = \frac{\delta^2 I_0(B)}{\delta B^2}, \quad S_0^{-1} = -\frac{\delta^2 I_0(c^*, c)}{\delta c^* \delta c}, \quad -\Sigma_{\text{mean}} = -\frac{\delta^2 I_{\text{int}}(c^*, c, B)}{\delta c^* \delta c} \quad (24)$$

$$\Gamma_2(G, S) = -\ln \frac{\int [dc^*][dc][dB] \exp \left\{ -\left[\frac{1}{2} B G^{-1} B + c^* S^{-1} c + I_{\text{int}}(c^*, c, B) \right] \right\}}{\int [dc^*][dc][dB] \exp \left\{ -\left[\frac{1}{2} B G^{-1} B + c^* S^{-1} c \right] \right\}} \Bigg|_{2\text{PI}} \quad (25)$$

where Σ_{mean} is the mean field contribution to the fermion self-energy; Γ_2 sums all 2PI diagrams. The prominent difference between these diagrams and the

perturbative ones is that the propagators constituting these diagrams are self-consistent ones G and S , not G_0 and S_0 . But the vertices are bare, which are same as those in perturbative diagrams.

3. Self-Energy of Fermion and Boson Fields

Γ_2 in Eq. (25) receives contributions from infinite 2PI diagrams, some of which are shown in Fig. 1. In actual calculations, it is impossible to sum all these diagrams. We have to truncate the effective action to certain order (But it is possible to resum some kinds of diagrams to infinite order through other methods, see Ref. [17] for more details). In this work we will truncate the effective action to three loops, i.e., only the first two diagram in Fig. 1 are employed in our calculations.

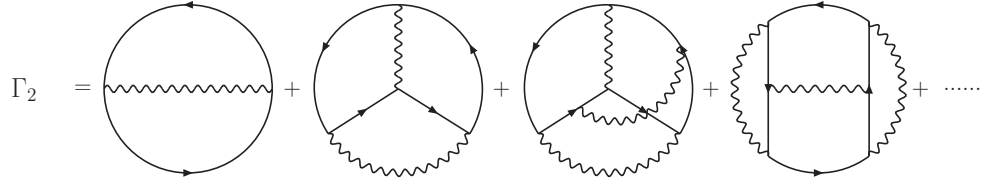


Figure 1: Two-particle irreducible vacuum diagrams contributing to the effective action, where the solid and wavy lines represent fermion and boson propagators, respectively.

In the following we will call the two-loop and three-loop diagrams as the LO and NLO contributions to Γ_2 , respectively. Their expressions are

$$\begin{aligned}\Gamma_2^{\text{LO}} &= \int_0^\beta d\tau_1 d\tau_2 \sum_{i_1 i_2} (-1)^{i_1+i_2} \frac{1}{2} U^2 G_{\tau_1 i_1, \tau_2 i_2} \text{tr}(\sigma^z S_{\tau_2 i_2, \tau_1 i_1} \sigma^z S_{\tau_1 i_1, \tau_2 i_2}), \\ \Gamma_2^{\text{NLO}} &= \int_0^\beta d\tau_1 d\tau_2 d\tau_3 d\tau_4 \sum_{i_1 i_2 i_3 i_4} (-1)^{i_1+i_2+i_3+i_4} \frac{1}{4} U^4 G_{\tau_1 i_1, \tau_3 i_3} G_{\tau_2 i_2, \tau_4 i_4} \\ &\quad \times \text{tr}(\sigma^z S_{\tau_4 i_4, \tau_1 i_1} \sigma^z S_{\tau_1 i_1, \tau_2 i_2} \sigma^z S_{\tau_2 i_2, \tau_3 i_3} \sigma^z S_{\tau_3 i_3, \tau_4 i_4}),\end{aligned}\quad (27)$$

where the trace tr only operates in spin space. Up to now, we have expressed the effective action as a functional of the self-consistent propagators G and S . Then one can employ Eq. (21) to obtain the self-consistent equations, given by

$$S^{-1} = S_0^{-1} - \Sigma, \quad (28)$$

$$G^{-1} = G_0^{-1} - \Pi, \quad (29)$$

where the fermion and boson self-energies are

$$\begin{aligned}\Sigma &= \frac{\delta\Gamma_{\text{int}}}{\delta S} = \Sigma_{\text{mean}} + \frac{\delta\Gamma_2(G, S)}{\delta S} \\ &\approx \Sigma_{\text{mean}} + \Sigma^{\text{LO}}(G, S) + \Sigma^{\text{NLO}}(G, S),\end{aligned}\quad (30)$$

$$\begin{aligned}\Pi &= -2\frac{\delta\Gamma_{\text{int}}}{\delta G} = -2\frac{\delta\Gamma_2(G, S)}{\delta G} \\ &\approx \Pi^{\text{LO}}(G, S) + \Pi^{\text{NLO}}(G, S).\end{aligned}\quad (31)$$

They are depicted in Fig. 2 and Fig. 3, respectively, whose expressions read

$$\Sigma_{\tau_1 i_1 \alpha_1, \tau_2 i_2 \alpha_2}^{\text{LO}} = \sum_{\beta_1 \beta_2} (-1)^{i_1+i_2} U^2 G_{\tau_1 i_1, \tau_2 i_2} \sigma_{\alpha_1 \beta_1}^z S_{\tau_1 i_1 \beta_1, \tau_2 i_2 \beta_2} \sigma_{\beta_2 \alpha_2}^z, \quad (32)$$

$$\begin{aligned}\Sigma_{\tau_1 i_1 \alpha_1, \tau_2 i_2 \alpha_2}^{\text{NLO}} &= \int_0^\beta d\tau_3 d\tau_4 \sum_{i_3 i_4} \sum_{\alpha_3 \alpha_4} \sum_{\beta_1 \dots \beta_4} (-1)^{i_1+i_2+i_3+i_4} U^4 G_{\tau_1 i_1, \tau_3 i_3} \\ &\quad \times G_{\tau_4 i_4, \tau_2 i_2} \sigma_{\alpha_1 \beta_1}^z S_{\tau_1 i_1 \beta_1, \tau_4 i_4 \alpha_4} \sigma_{\alpha_4 \beta_4}^z S_{\tau_4 i_4 \beta_4, \tau_3 i_3 \alpha_3} \sigma_{\alpha_3 \beta_3}^z \\ &\quad \times S_{\tau_3 i_3 \beta_3, \tau_2 i_2 \beta_2} \sigma_{\beta_2 \alpha_2}^z, \end{aligned}\quad (33)$$

$$\Pi_{\tau_1 i_1, \tau_2 i_2}^{\text{LO}} = -U^2 (-1)^{i_1+i_2} \text{tr}(\sigma^z S_{\tau_1 i_1, \tau_2 i_2} \sigma^z S_{\tau_2 i_2, \tau_1 i_1}), \quad (34)$$

$$\begin{aligned}\Pi_{\tau_1 i_1, \tau_2 i_2}^{\text{NLO}} &= \int_0^\beta d\tau_3 d\tau_4 \sum_{i_3 i_4} (-U^4) G_{\tau_3 i_3, \tau_4 i_4} (-1)^{i_1+i_2+i_3+i_4} \\ &\quad \times \text{tr}(\sigma^z S_{\tau_1 i_1, \tau_4 i_4} \sigma^z S_{\tau_4 i_4, \tau_2 i_2} \sigma^z S_{\tau_2 i_2, \tau_3 i_3} \sigma^z S_{\tau_3 i_3, \tau_1 i_1}). \end{aligned}\quad (35)$$

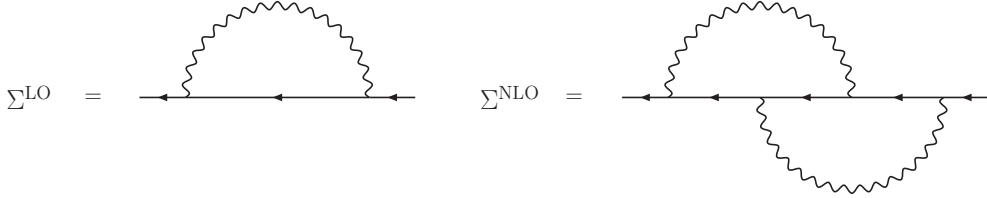


Figure 2: LO and NLO contributions to the fermion self-energy.

Up to now, we have worked in coordinate space. In fact, it is more convenient to employ the momentum lattices, in which numerical calculations are easier. The coordinate lattices and the momentum ones are related through the Fourier transformation. For example, for the boson and fermion propagators we have

$$G_{\tau_1 i_1, \tau_2 i_2} = \beta^{-2} \frac{1}{(\sqrt{N})^2} \sum_{\omega_1 k_1} \sum_{\omega_2 k_2} G(\omega_1 k_1, \omega_2 k_2) \exp(-i\omega_1 \tau_1 + i\omega_2 \tau_2$$

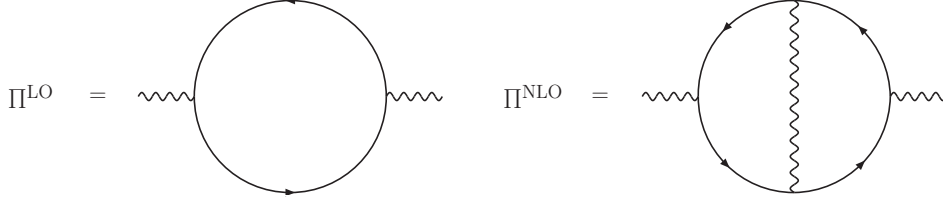


Figure 3: LO and NLO contributions to the boson self-energy.

$$+ik_1 R_{i_1} - ik_2 R_{i_2}), \quad (36)$$

$$S_{\tau_1 i_1 \alpha_1, \tau_2 i_2 \alpha_2} = \beta^{-2} \frac{1}{(\sqrt{N})^2} \sum_{\omega_1 k_1} \sum_{\omega_2 k_2} S_{\alpha_1, \alpha_2}(\omega_1 k_1, \omega_2 k_2) \times \exp(-i\omega_1 \tau_1 + i\omega_2 \tau_2 + ik_1 R_{i_1} - ik_2 R_{i_2}), \quad (37)$$

where N is the number of lattice sites, momentum sums are restricted to the Brillouin zone, and the Matsubara frequencies are

$$\omega_i = \begin{cases} (2n_i + 1)\pi T & \text{(Fermion)} \\ 2n_i \pi T & \text{(Boson)} \end{cases}. \quad (38)$$

Employing Eqs. (36) and (37), we can reexpress the self-energies in Eqs. (32)—(35) in momentum lattices as follow

$$\begin{aligned} \Sigma_{\alpha_1, \alpha_2}^{\text{LO}}(\bar{\omega}_1 \bar{k}_1, \bar{\omega}_2 \bar{k}_2) &= \beta^{-2} \frac{1}{(\sqrt{N})^2} \sum_{\omega_1 k_1} \sum_{\omega_2 k_2} \sum_{\alpha'_1 \alpha'_2} U^2 G(\omega_1 k_1, \omega_2 k_2) \sigma_{\alpha_1 \alpha'_1}^z \\ &\times S_{\alpha'_1, \alpha'_2}(\bar{\omega}_1 - \omega_1, \bar{k}_1 - k_1 - Q; \bar{\omega}_2 - \omega_2, \bar{k}_2 - k_2 - Q) \\ &\times \sigma_{\alpha'_2 \alpha_2}^z e^{-i(\bar{\omega}_1 - \omega_1)0^+}, \end{aligned} \quad (39)$$

$$\begin{aligned} \Sigma_{\alpha_1, \alpha_2}^{\text{NLO}}(\bar{\omega}_1 \bar{k}_1, \bar{\omega}_2 \bar{k}_2) &= \beta^{-6} \frac{1}{(\sqrt{N})^4} \sum_{\omega_1 k_1 \dots \omega_4 k_4} \sum_{\omega'_3 k'_3} \sum_{\omega'_4 k'_4} \sum_{\alpha_3 \alpha_4} \sum_{\alpha'_1 \dots \alpha'_4} U^4 \\ &\times G(\omega_1 k_1, \omega_4 k_4) G(\omega_3 k_3, \omega_2 k_2) \\ &\times \sigma_{\alpha_1 \alpha'_1}^z S_{\alpha'_1, \alpha'_3}(\bar{\omega}_1 - \omega_1, \bar{k}_1 - k_1 - Q; \omega'_3, k'_3) \sigma_{\alpha'_3 \alpha_3}^z \\ &\times S_{\alpha_3, \alpha_4}(\omega'_3 - \omega_3, k'_3 - k_3 - Q; \omega'_4 - \omega_4, k'_4 - k_4 - Q) \\ &\times \sigma_{\alpha_4 \alpha'_4}^z S_{\alpha'_4, \alpha'_2}(\omega'_4, k'_4; \bar{\omega}_2 - \omega_2, \bar{k}_2 - k_2 - Q) \sigma_{\alpha'_2 \alpha_2}^z, \end{aligned} \quad (40)$$

$$\begin{aligned} \Pi^{\text{LO}}(\bar{\omega}_1 \bar{k}_1, \bar{\omega}_2 \bar{k}_2) &= -\beta^{-2} \frac{1}{(\sqrt{N})^2} \sum_{\omega_1 k_1} \sum_{\omega_2 k_2} U^2 \text{tr} \left[S(\omega_1 k_1, \omega_2 k_2) \sigma^z \right. \\ &\times S(\omega_2 - \bar{\omega}_2, k_2 - \bar{k}_2 + Q; \omega_1 - \bar{\omega}_1, k_1 - \bar{k}_1 + Q) \sigma^z \left. \right], \end{aligned} \quad (41)$$

$$\begin{aligned}
\Pi^{\text{NLO}}(\bar{\omega}_1 \bar{k}_1, \bar{\omega}_2 \bar{k}_2) = & -\beta^{-6} \frac{1}{(\sqrt{N})^4} \sum_{\omega_1 k_1 \dots \omega_4 k_4} \sum_{\omega'_3 k'_3} \sum_{\omega'_4 k'_4} U^4 \\
& \times G(\omega_3 - \omega'_3, k_3 - k'_3 - Q; \omega'_4 - \omega_4, k'_4 - k_4 - Q) \\
& \times \text{tr} \left[S(\omega_1 k_1, \omega_4 k_4) \sigma^z S(\omega'_4, k'_4; \omega_2 + \bar{\omega}_2, k_2 + \bar{k}_2 - Q) \right. \\
& \times \sigma^z S(\omega_2 k_2, \omega_3 k_3) \sigma^z \\
& \left. \times S(\omega'_3, k'_3; \omega_1 - \bar{\omega}_1, k_1 - \bar{k}_1 + Q) \sigma^z \right], \tag{42}
\end{aligned}$$

where $Q = (\pi, \pi)^T$ (in units of the inverse lattice spacing) is the nesting vector in two dimensions.

In order to simplify our calculations, we assume that the anti-ferromagnetic order parameter B in Eq. (22) is constant. Then the inverse of the fermion propagator can be expressed as the following formalism:

$$S_{\alpha_1 \alpha_2}^{-1}(\omega_1 k_1, \omega_2 k_2) = \beta C(\omega_1, k_1) \delta_{\omega_1 \omega_2} \delta_{k_1 k_2} \delta_{\alpha_1 \alpha_2} + \beta D(\omega_1, k_1) \delta_{\omega_1 \omega_2} \delta_{k_1, k_2 + Q} \sigma_{\alpha_1 \alpha_2}^z. \tag{43}$$

It then follows that

$$S_{\alpha_1 \alpha_2}(\omega_1 k_1, \omega_2 k_2) = \beta \bar{C}(\omega_1, k_1) \delta_{\omega_1 \omega_2} \delta_{k_1 k_2} \delta_{\alpha_1 \alpha_2} + \beta \bar{D}(\omega_1, k_1) \delta_{\omega_1 \omega_2} \delta_{k_1, k_2 + Q} \sigma_{\alpha_1 \alpha_2}^z, \tag{44}$$

with

$$\bar{C}(\omega_1, k_1) = \frac{C(\omega_1, k_1 + Q)}{C(\omega_1, k_1)C(\omega_1, k_1 + Q) - D(\omega_1, k_1)D(\omega_1, k_1 + Q)}, \tag{45}$$

$$\bar{D}(\omega_1, k_1) = \frac{-D(\omega_1, k_1)}{C(\omega_1, k_1)C(\omega_1, k_1 + Q) - D(\omega_1, k_1)D(\omega_1, k_1 + Q)}. \tag{46}$$

Similarly, for the boson field one has

$$G^{-1}(\omega_1 k_1, \omega_2 k_2) = \beta A(\omega_1, k_1) \delta_{\omega_1 \omega_2} \delta_{k_1 k_2}, \tag{47}$$

$$G(\omega_1 k_1, \omega_2 k_2) = \beta A^{-1}(\omega_1, k_1) \delta_{\omega_1 \omega_2} \delta_{k_1 k_2}. \tag{48}$$

Substituting Eqs. (44) and (48) into Eqs. (39)—(42), we obtain

$$\begin{aligned}
\Sigma_{\alpha_1, \alpha_2}^{\text{LO}}(\bar{\omega}_1 \bar{k}_1, \bar{\omega}_2 \bar{k}_2) = & \Sigma_+^{\text{LO}}(\bar{\omega}_1, \bar{k}_1) \delta_{\bar{\omega}_1 \bar{\omega}_2} \delta_{\bar{k}_1 \bar{k}_2} \delta_{\alpha_1 \alpha_2} \\
& + \Sigma_-^{\text{LO}}(\bar{\omega}_1, \bar{k}_1) \delta_{\bar{\omega}_1 \bar{\omega}_2} \delta_{\bar{k}_1, \bar{k}_2 + Q} \sigma_{\alpha_1 \alpha_2}^z, \tag{49}
\end{aligned}$$

$$\begin{aligned}
\Sigma_{\alpha_1, \alpha_2}^{\text{NLO}}(\bar{\omega}_1 \bar{k}_1, \bar{\omega}_2 \bar{k}_2) = & \Sigma_+^{\text{NLO}}(\bar{\omega}_1, \bar{k}_1) \delta_{\bar{\omega}_1 \bar{\omega}_2} \delta_{\bar{k}_1 \bar{k}_2} \delta_{\alpha_1 \alpha_2} \\
& + \Sigma_-^{\text{NLO}}(\bar{\omega}_1, \bar{k}_1) \delta_{\bar{\omega}_1 \bar{\omega}_2} \delta_{\bar{k}_1, \bar{k}_2 + Q} \sigma_{\alpha_1 \alpha_2}^z, \tag{50}
\end{aligned}$$

$$\Pi^{\text{LO}}(\bar{\omega}_1 \bar{k}_1, \bar{\omega}_2 \bar{k}_2) = \Pi^{\text{LO}}(\bar{\omega}_1, \bar{k}_1) \delta_{\bar{\omega}_1 \bar{\omega}_2} \delta_{\bar{k}_1 \bar{k}_2}, \tag{51}$$

$$\Pi^{\text{NLO}}(\bar{\omega}_1 \bar{k}_1, \bar{\omega}_2 \bar{k}_2) = \Pi^{\text{NLO}}(\bar{\omega}_1, \bar{k}_1) \delta_{\bar{\omega}_1 \bar{\omega}_2} \delta_{\bar{k}_1 \bar{k}_2}, \tag{52}$$

with

$$\begin{aligned}\Sigma_+^{\text{LO}}(\bar{\omega}_1, \bar{k}_1) &= U^2 \frac{1}{(\sqrt{N})^2} \sum_{\omega_1 k_1} A^{-1}(\omega_1, k_1) \\ &\quad \times \bar{C}(\bar{\omega}_1 - \omega_1, \bar{k}_1 - k_1 - Q) e^{-i(\bar{\omega}_1 - \omega_1)0^+},\end{aligned}\quad (53)$$

$$\begin{aligned}\Sigma_-^{\text{LO}}(\bar{\omega}_1, \bar{k}_1) &= U^2 \frac{1}{(\sqrt{N})^2} \sum_{\omega_1 k_1} A^{-1}(\omega_1, k_1) \\ &\quad \times \bar{D}(\bar{\omega}_1 - \omega_1, \bar{k}_1 - k_1 - Q) e^{-i(\bar{\omega}_1 - \omega_1)0^+},\end{aligned}\quad (54)$$

$$\begin{aligned}\Sigma_+^{\text{NLO}}(\bar{\omega}_1, \bar{k}_1) &= U^4 \beta^{-1} \frac{1}{(\sqrt{N})^4} \sum_{\omega_1 k_1} \sum_{\omega_2 k_2} A^{-1}(\omega_1, k_1) A^{-1}(\omega_2, k_2) \\ &\quad \times \left[\bar{C}(\bar{\omega}_1 - \omega_1, \bar{k}_1 - k_1 - Q) \bar{C}(\bar{\omega}_1 - \omega_1 - \omega_2, \bar{k}_1 - k_1 - k_2) \right. \\ &\quad \times \bar{C}(\bar{\omega}_1 - \omega_2, \bar{k}_1 - k_2 - Q) + \bar{C}(\bar{\omega}_1 - \omega_1, \bar{k}_1 - k_1 - Q) \\ &\quad \times \bar{D}(\bar{\omega}_1 - \omega_1 - \omega_2, \bar{k}_1 - k_1 - k_2) \bar{D}(\bar{\omega}_1 - \omega_2, \bar{k}_1 - k_2) \\ &\quad + \bar{D}(\bar{\omega}_1 - \omega_1, \bar{k}_1 - k_1 - Q) \\ &\quad \times \bar{C}(\bar{\omega}_1 - \omega_1 - \omega_2, \bar{k}_1 - k_1 - k_2 - Q) \bar{D}(\bar{\omega}_1 - \omega_2, \bar{k}_1 - k_2) \\ &\quad + \bar{D}(\bar{\omega}_1 - \omega_1, \bar{k}_1 - k_1 - Q) \\ &\quad \times \bar{D}(\bar{\omega}_1 - \omega_1 - \omega_2, \bar{k}_1 - k_1 - k_2 - Q) \\ &\quad \left. \times \bar{C}(\bar{\omega}_1 - \omega_2, \bar{k}_1 - k_2 - Q) \right],\end{aligned}\quad (55)$$

$$\begin{aligned}\Sigma_-^{\text{NLO}}(\bar{\omega}_1, \bar{k}_1) &= U^4 \beta^{-1} \frac{1}{(\sqrt{N})^4} \sum_{\omega_1 k_1} \sum_{\omega_2 k_2} A^{-1}(\omega_1, k_1) A^{-1}(\omega_2, k_2) \\ &\quad \times \left[\bar{C}(\bar{\omega}_1 - \omega_1, \bar{k}_1 - k_1 - Q) \bar{C}(\bar{\omega}_1 - \omega_1 - \omega_2, \bar{k}_1 - k_1 - k_2) \right. \\ &\quad \times \bar{D}(\bar{\omega}_1 - \omega_2, \bar{k}_1 - k_2 - Q) + \bar{C}(\bar{\omega}_1 - \omega_1, \bar{k}_1 - k_1 - Q) \\ &\quad \times \bar{D}(\bar{\omega}_1 - \omega_1 - \omega_2, \bar{k}_1 - k_1 - k_2) \bar{C}(\bar{\omega}_1 - \omega_2, \bar{k}_1 - k_2) \\ &\quad + \bar{D}(\bar{\omega}_1 - \omega_1, \bar{k}_1 - k_1 - Q) \\ &\quad \times \bar{C}(\bar{\omega}_1 - \omega_1 - \omega_2, \bar{k}_1 - k_1 - k_2 - Q) \bar{C}(\bar{\omega}_1 - \omega_2, \bar{k}_1 - k_2) \\ &\quad + \bar{D}(\bar{\omega}_1 - \omega_1, \bar{k}_1 - k_1 - Q) \\ &\quad \times \bar{D}(\bar{\omega}_1 - \omega_1 - \omega_2, \bar{k}_1 - k_1 - k_2 - Q) \\ &\quad \left. \times \bar{D}(\bar{\omega}_1 - \omega_2, \bar{k}_1 - k_2 - Q) \right],\end{aligned}\quad (56)$$

and

$$\begin{aligned}\Pi^{\text{LO}}(\bar{\omega}_1, \bar{k}_1) &= -2U^2 \frac{1}{(\sqrt{N})^2} \sum_{\omega_1 k_1} \left[\bar{C}(\omega_1, k_1) \bar{C}(\omega_1 - \bar{\omega}_1, k_1 - \bar{k}_1 + Q) \right. \\ &\quad \left. + \bar{D}(\omega_1, k_1) \bar{D}(\omega_1 - \bar{\omega}_1, k_1 - \bar{k}_1) \right],\end{aligned}\quad (57)$$

$$\begin{aligned}\Pi^{\text{NLO}}(\bar{\omega}_1, \bar{k}_1) &= -2U^4 \beta^{-1} \frac{1}{(\sqrt{N})^4} \sum_{\omega_1 k_1} \sum_{\omega_2 k_2} \left[A^{-1}(\omega_2 - \omega_1 + \bar{\omega}_1, k_2 - k_1 + \bar{k}_1) \right. \\ &\quad \times \bar{C}(\omega_1, k_1) \bar{C}(\omega_2 + \bar{\omega}_1, k_2 + \bar{k}_1 - Q) \bar{C}(\omega_2, k_2) \\ &\quad \times \bar{C}(\omega_1 - \bar{\omega}_1, k_1 - \bar{k}_1 + Q) \\ &\quad + A^{-1}(\omega_2 - \omega_1 + \bar{\omega}_1, k_2 - k_1 + \bar{k}_1) \bar{C}(\omega_1, k_1) \\ &\quad \times \bar{C}(\omega_2 + \bar{\omega}_1, k_2 + \bar{k}_1 - Q) \bar{D}(\omega_2, k_2) \bar{D}(\omega_1 - \bar{\omega}_1, k_1 - \bar{k}_1) \\ &\quad + A^{-1}(\omega_2 - \omega_1 + \bar{\omega}_1, k_2 - k_1 + \bar{k}_1) \bar{D}(\omega_1, k_1) \\ &\quad \times \bar{D}(\omega_2 + \bar{\omega}_1, k_2 + \bar{k}_1) \bar{C}(\omega_2, k_2) \bar{C}(\omega_1 - \bar{\omega}_1, k_1 - \bar{k}_1 + Q) \\ &\quad + A^{-1}(\omega_2 - \omega_1 + \bar{\omega}_1, k_2 - k_1 + \bar{k}_1) \bar{D}(\omega_1, k_1) \\ &\quad \times \bar{D}(\omega_2 + \bar{\omega}_1, k_2 + \bar{k}_1) \bar{D}(\omega_2, k_2) \bar{D}(\omega_1 - \bar{\omega}_1, k_1 - \bar{k}_1) \\ &\quad + A^{-1}(\omega_2 - \omega_1 + \bar{\omega}_1, k_2 - k_1 + \bar{k}_1 - Q) \bar{C}(\omega_1, k_1) \\ &\quad \times \bar{D}(\omega_2 + \bar{\omega}_1, k_2 + \bar{k}_1) \bar{C}(\omega_2, k_2) \bar{D}(\omega_1 - \bar{\omega}_1, k_1 - \bar{k}_1) \\ &\quad + A^{-1}(\omega_2 - \omega_1 + \bar{\omega}_1, k_2 - k_1 + \bar{k}_1 - Q) \bar{C}(\omega_1, k_1) \\ &\quad \times \bar{D}(\omega_2 + \bar{\omega}_1, k_2 + \bar{k}_1) \bar{D}(\omega_2, k_2) \bar{C}(\omega_1 - \bar{\omega}_1, k_1 - \bar{k}_1 + Q) \\ &\quad + A^{-1}(\omega_2 - \omega_1 + \bar{\omega}_1, k_2 - k_1 + \bar{k}_1 - Q) \bar{D}(\omega_1, k_1) \\ &\quad \times \bar{C}(\omega_2 + \bar{\omega}_1, k_2 + \bar{k}_1 - Q) \bar{C}(\omega_2, k_2) \bar{D}(\omega_1 - \bar{\omega}_1, k_1 - \bar{k}_1) \\ &\quad + A^{-1}(\omega_2 - \omega_1 + \bar{\omega}_1, k_2 - k_1 + \bar{k}_1 - Q) \bar{D}(\omega_1, k_1) \\ &\quad \times \bar{C}(\omega_2 + \bar{\omega}_1, k_2 + \bar{k}_1 - Q) \\ &\quad \left. \times \bar{D}(\omega_2, k_2) \bar{C}(\omega_1 - \bar{\omega}_1, k_1 - \bar{k}_1 + Q) \right].\end{aligned}\quad (58)$$

Furthermore, from Eq. (24) it is easy to obtain

$$G_0^{-1}(\omega_1 k_1, \omega_2 k_2) = \beta U \delta_{\omega_1 \omega_2} \delta_{k_1 k_2}, \quad (59)$$

$$S_{0; \alpha_1 \alpha_2}^{-1}(\omega_1 k_1, \omega_2 k_2) = \beta (-i\omega_1 + \xi_{k_1}) \delta_{\omega_1 \omega_2} \delta_{k_1 k_2} \delta_{\alpha_1 \alpha_2}, \quad (60)$$

$$\Sigma_{\text{mean}; \alpha_1, \alpha_2}(\omega_1 k_1, \omega_2 k_2) = \beta U B \delta_{\omega_1 \omega_2} \delta_{k_1, k_2 + Q} \sigma_{\alpha_1 \alpha_2}^z, \quad (61)$$

with

$$\xi_k = -2t(\cos k_x + \cos k_y) - \mu, \quad (62)$$

where the momenta k_x and k_y are in units of the inverse lattice spacing. Substituting Eqs. (49)—(52) and Eqs. (59)—(61) into the self-consistent equations (28) and (29), we finally get

$$A(\omega, k) = U - \beta^{-1}\Pi^{\text{LO}}(\omega, k) - \beta^{-1}\Pi^{\text{NLO}}(\omega, k), \quad (63)$$

$$C(\omega, k) = -i\omega + \xi_k - \beta^{-1}\Sigma_+^{\text{LO}}(\omega, k) - \beta^{-1}\Sigma_+^{\text{NLO}}(\omega, k), \quad (64)$$

$$D(\omega, k) = -UB - \beta^{-1}\Sigma_-^{\text{LO}}(\omega, k) - \beta^{-1}\Sigma_-^{\text{NLO}}(\omega, k). \quad (65)$$

4. Numerical Results

Besides equations (63)—(65), we still need another two equations to perform the numerical calculations. One is the self-consistent equation for the antiferromagnetic order parameter B , which can be obtained from the first equation in Eqs. (20), as given by

$$B = -2\beta^{-1} \frac{1}{(\sqrt{N})^2} \sum_{\omega k} \bar{D}(\omega, k); \quad (66)$$

the other one is the equation for the electron density (number per lattice site):

$$n = -2\beta^{-1} \frac{1}{(\sqrt{N})^2} \sum_{\omega k} \bar{C}(\omega, k) e^{i\omega 0^+}. \quad (67)$$

We employ a $N_s \times N_s \times N_\tau$ lattice, where N_s corresponds to the lattice number in one spatial dimension and we have two-dimensional lattice number $N = N_s \times N_s$; N_τ is the temporal lattice number. The calculations are performed in the momentum and frequency $(k_x, k_y, i\omega)$ space, which are equivalent to those in the coordinate space with spacial periodic boundary conditions and temporal anti-periodic (periodic) boundary conditions for fermion (boson) fields.

In order to save computing time, we also employ the symmetries of the lattice to simplify our calculations. Propagators and self-energies are invariant under the following symmetry operations:

$$k_x \leftrightarrow k_y, \quad k_x \rightarrow -k_x, \quad k_y \rightarrow -k_y. \quad (68)$$

Therefore, we only need to compute one-eighth of lattice points in the Brillouin zone. Furthermore, we also have the time-reversal symmetry as follows

$$A(-\omega, k) = A(\omega, k)^*, \quad (69)$$

$$C(-\omega, k) = C(\omega, k)^*, \quad (70)$$

$$D(-\omega, k) = D(\omega, k)^*. \quad (71)$$

For this reason, it is only necessary to calculate the propagators when $\omega \geq 0$.

In this work, we want to investigate the importance of higher-order fluctuations in the 2PI effective action theory. In another word, we would like to study whether the expansions of the effective action in order of loops are convergent, when the interaction strength U becomes large. In order to obtain this goal, we perform three different calculations: the first one is just the mean field calculation, i.e., neglecting the influence of the fluctuations and assuming $\Gamma_2 = 0$. The second one is that only the two-loop contribution (the first diagram in Fig. 1) to Γ_2 are included. In this calculation, equations (63)—(67) constitute a closed system and are solved self-consistently through iterations, but with $\Pi^{\text{NLO}} = \Sigma_+^{\text{NLO}} = \Sigma_-^{\text{NLO}} = 0$. As for the third calculation, it should have been solving this set of equations self-consistently with $\Pi^{\text{NLO}} = \Sigma_+^{\text{NLO}} = \Sigma_-^{\text{NLO}} \neq 0$, but such self-consistent calculation need lots of computing time and is beyond our computing abilities. Therefore, in the third calculation we substitute the results of the self-consistent equations which only include the LO contribution to the fluctuations, i.e., the results obtained from the second calculation, into equations (55), (56), and (58) to obtain the NLO self energies. We should emphasize that this calculation is not self-consistent, but it is still reasonable to get some information on the convergence of the theory from this calculation.

Equations (63)—(67) are iterated successively to search for self-consistent solutions. In order to avoid oscillations during the calculations, we also employ feedback in the process of iterations, i.e., the updated value is chosen to be a weighted average of the new calculated one and the old one in every iteration. We use the following criteria to judge whether the convergence of iterations is achieved:

$$\frac{|A_{\text{new}}(\omega, k) - A_{\text{old}}(\omega, k)|}{|A_{\text{new}}(\omega, k)|} < 0.001, \quad (72)$$

$$\frac{|C_{\text{new}}(\omega, k) - C_{\text{old}}(\omega, k)|}{|C_{\text{new}}(\omega, k)|} < 0.001, \quad (73)$$

$$\frac{|D_{\text{new}}(\omega, k) - D_{\text{old}}(\omega, k)|}{|D_{\text{new}}(\omega, k)|} < 0.001, \quad (74)$$

for all ω and k , and

$$\frac{|B_{\text{new}} - B_{\text{old}}|}{|B_{\text{new}}|} < 0.001, \quad (75)$$

$$\frac{|\mu_{\text{new}} - \mu_{\text{old}}|}{|\mu_{\text{new}}|} < 0.001, \quad (76)$$

where the update of the chemical potential μ is realized through a Newton's procedure at a fixed electron density n . Furthermore, we would like to emphasize that there is an exponential factor appearing in Eqs. (53), (54), and (67). This exponential factor contains an infinitesimal positive constant 0^+ . In the numerical calculations, this constant can not be chosen to be very small, because that would affect the convergence of the frequency summation, since the infinite frequency summation is cut to be finite in numerical calculations. Therefore, in our work we choose a relatively small constant, and we also check that the numerical results are insensitive to the choice of this constant.

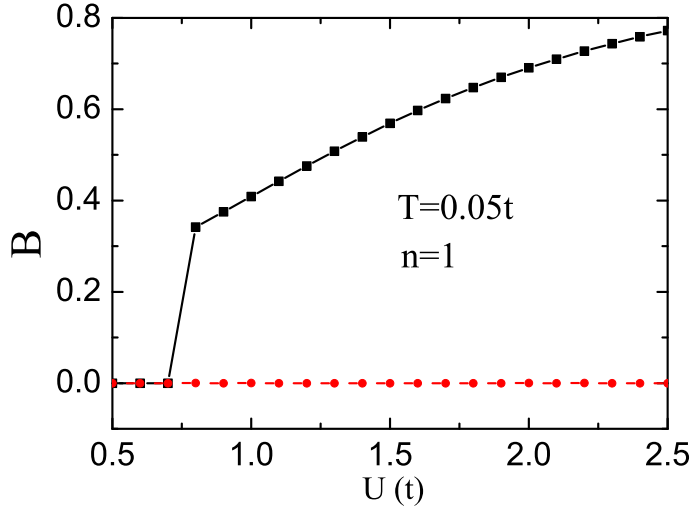


Figure 4: (color online). Antiferromagnetic order parameter B as a function of the Coulomb repulsion U in unit of the hopping t . Here the temperature is chosen to be $T = 0.05t$ and the electron density $n = 1$. The black solid line with squares corresponds to the mean field calculations and the red dashed line with circles to the self-consistent calculations including LO fluctuations. For the self-consistent calculations, the Brillouin zone is discretized into a 10×10 lattice and the temporal lattice number is chosen to be $N_\tau = 64$.

The antiferromagnetic order parameter B as a function of U , obtained in the mean field approximations and the self-consistent calculations including

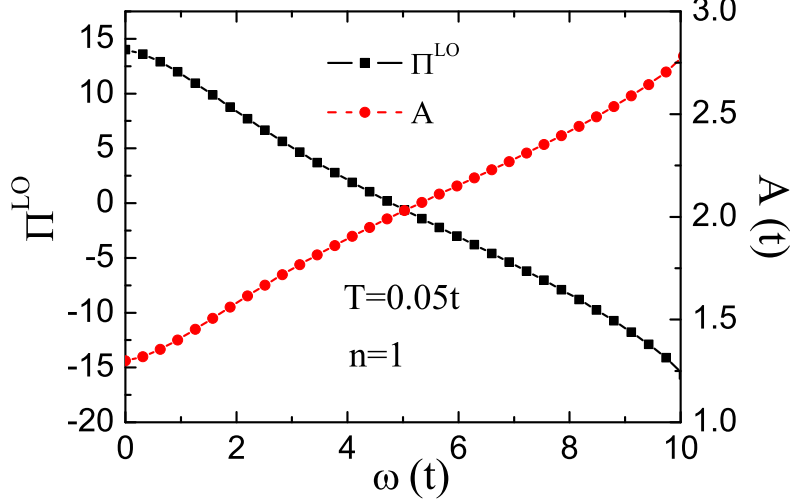


Figure 5: (color online). Dependence of the LO boson self-energy $\Pi^{\text{LO}}(\omega, k_x = 0, k_y = 0)$ and the inverse of the boson propagator $A(\omega, k_x = 0, k_y = 0)$ on the Matsubara frequencies, obtained in the self-consistent calculations with LO fluctuations included. The Coulomb repulsion is $U = 2t$. These results are based on a lattice of $10 \times 10 \times 64$.

LO fluctuations, are shown in Fig. 4. In the mean field approximations, the frequency summation in Eqs. (66) and (67) can be performed analytically. One can see that B develops a nonvanishing value with the increase of the Coulomb repulsion in the mean field approximations, which means that a phase transition occurs and a long-range antiferromagnetic state is formed. However, in the self-consistent calculations with LO fluctuations included, we do not find any phase transition at $T = 0.05t$, $n = 1$, and $U = 0.5 \sim 2.5t$. In order to find the reason why there is no phase transition in the self-consistent calculations, we show the LO boson self-energy and the inverse of the boson propagator A in Fig. 5. We can observe that Π^{LO} is positive at low frequencies, which results in that A is decreased at low frequencies compared to the bare Coulomb repulsion U . Therefore, the effective interaction is reduced by the fluctuations at low frequencies, which may account for the reason why the phase transition becomes difficult in the self-consistent calculations with fluctuations included.

Figure 6 shows the LO and NLO self-energies as functions of the on-site repulsion U . The boson self-energies Π are real and are presented in the two

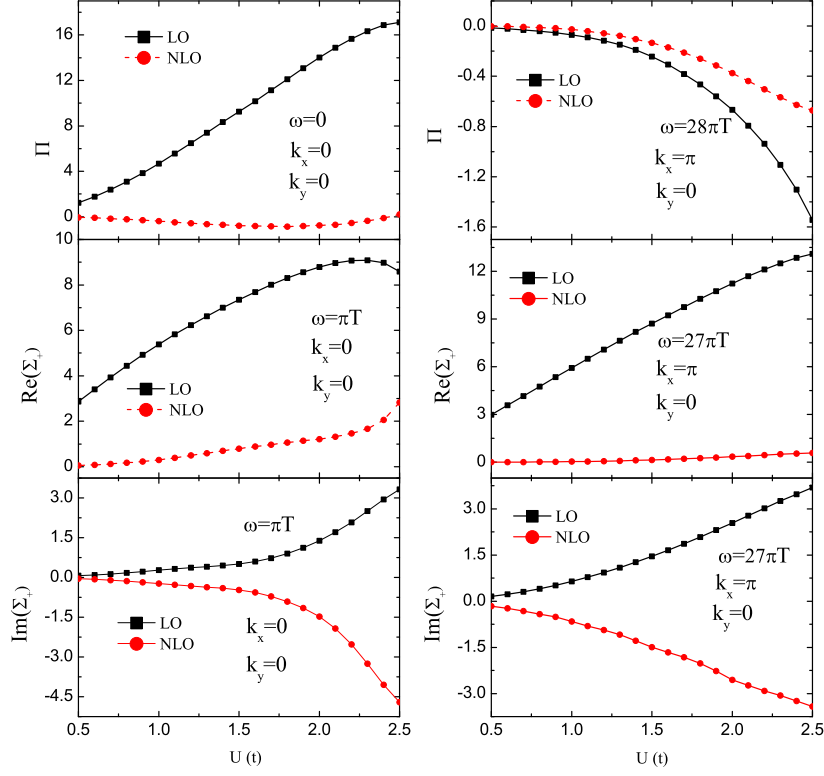


Figure 6: (color online). Comparison between the LO self-energies and the NLO ones. They are depicted as functions of the Coulomb repulsion U . Temperature $T = 0.05t$ and electron density $n = 1$ are chosen here. k_x and k_y are in unit of the inverse of the lattice spacing. These results are based on a lattice of $10 \times 10 \times 64$.

top panels. In the symmetrical phase where $B = 0$, Σ_- defined in Eqs. (49) and (50) is vanishing. Therefore, we only show Σ_+ here, whose real and imaginary parts are depicted in the two medium panels and the two bottom ones of Fig. 6, respectively. From this figure, one can observe that the NLO contributions to Π and the real part of Σ_+ are less than the LO ones, but as the U is larger than about $2t$, the NLO contributions can not be neglected as the top-right and medium-left panels show. However, for the imaginary part of the Σ_+ , we can see that the NLO contributions are completely comparable

with the LO ones, even when the U is less than $2t$.

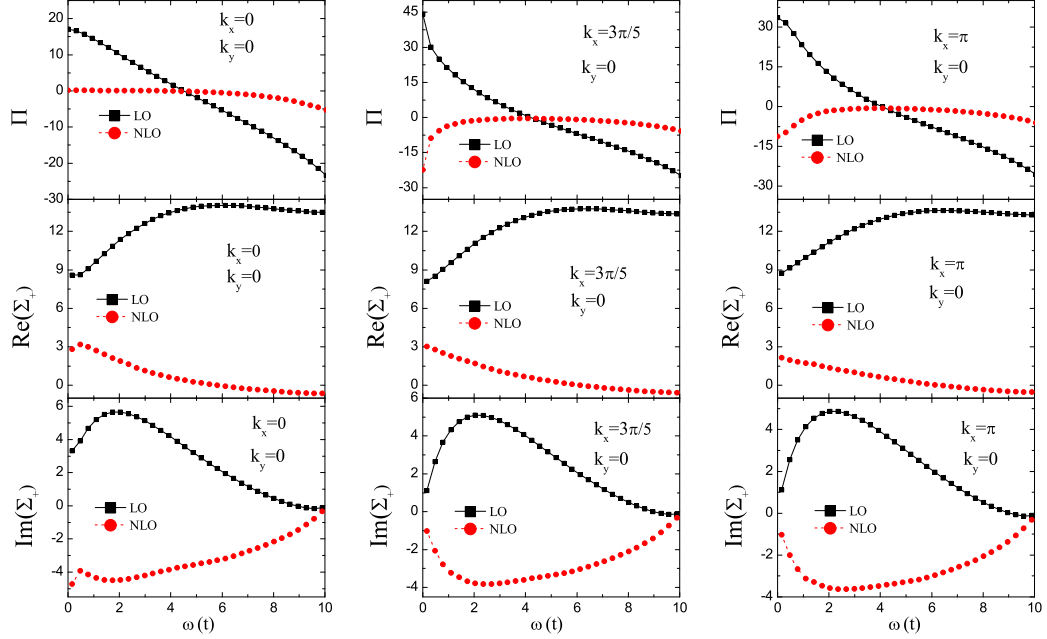


Figure 7: (color online). LO and NLO self-energies as functions of the frequencies. Three columns correspond to three different values of k_x . In the same way, we choose the temperature $T = 0.05t$, electron density $n = 1$, and the Coulomb repulsion $U = 2.5t$. The lattice is $10 \times 10 \times 64$.

In order to make the comparison much clearer, we show the spectrum of the self-energies in Fig. 7. Here we choose $U = 2.5t$. For the boson self-energies, we find that the NLO contributions are significant and should not be neglected, especially at low frequencies as the top-middle and top-right panels show. In the same way, the same conclusions are also appropriate for the real part of the Σ_+ . Furthermore, we also find that the real part of the NLO Σ_+ approaches zero at high frequencies. For the imaginary part of the Σ_+ , we confirm that the NLO contributions are comparable with the LO ones, which is clear at the whole region of the frequencies as the bottom three panels show.

Figure 8 depicts the self-energies at several values of the temperature.

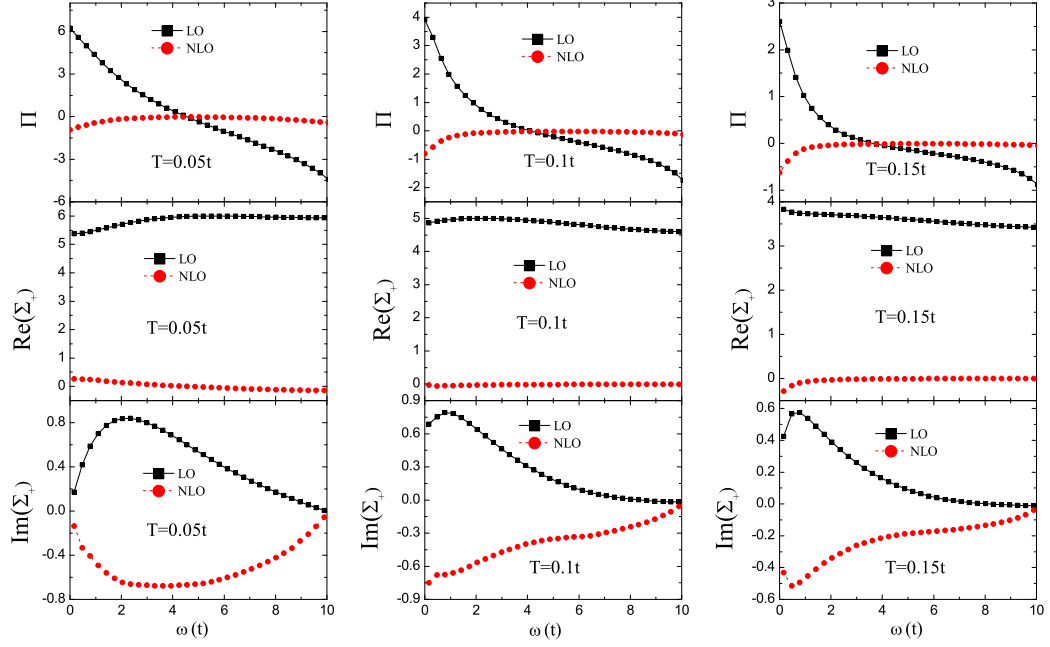


Figure 8: (color online). LO and NLO self-energies as functions of the frequencies at three different values of the temperature. We choose $k_x = 3\pi/5$, $k_y = 0$, $U = 1t$, and $n = 1$ in all these calculations. These results are based on a lattice of $10 \times 10 \times 64$.

Here we choose the Coulomb repulsion $U = t$. We have demonstrated above that at this value of U , the NLO contributions to the boson self-energy Π and the real part of the fermion self-energy Σ_+ are quite smaller than the LO ones, which are confirmed at other values of temperature as the first two rows of Fig. 8 show. In the same way, at other values of temperature, we also find that the NLO contributions to the imaginary part of the fermion self-energy are comparable to those of LO.

5. Summary and Conclusions

In this work, we have employed the 2PI effective action theories to study the strongly fluctuating electron systems, under the formalism of the two-dimensional Hubbard model. We first bosonize the original classic action of

the Hubbard model, then obtain the corresponding quantum 2PI effective action. In our actual calculations, the 2PI effective action is expanded to three loops. Therefore, LO and NLO quantum fluctuations are included in our approaches. We also obtain the LO and NLO contributions to the fermion and boson self-energies. They are expressed in the momentum-frequency space in a form which is very appropriate for numerical calculations.

We also perform numerical calculations on a lattice. Our numerical results indicate that due to the decrease of the effective Coulomb repulsion at low frequencies when the quantum fluctuations are included, the state with an antiferromagnetic long-range order is more difficult to formed. We also compare the LO and NLO contributions to the fermion and boson self-energies. We find that the NLO contributions to the boson self-energy and the real part of the fermion self-energy are less than the LO ones, but as the Coulomb repulsion energy U is larger than about $2t$, the NLO contributions can not be neglected. However, for the imaginary part of the fermion self-energy, it is found that the NLO contributions are comparable with the LO ones, even when the U is less than $2t$. However, their signs are opposite and their sum almost approaches zero.

Based on our calculations, we conclude that the 2PI effective action formalism, which is popularly employed in the particle physics and field theories, is also an appropriate approach to describe the strongly correlated electron systems. Higher-order quantum fluctuations are easily included in this approach. We must point out that one should be very careful about results obtained from the mean-field calculations of the strongly correlated electron systems, or even those including the LO quantum fluctuations. Because higher-order quantum fluctuations in the systems also play an important role, which can not be neglected.

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